Stability of a Spinning Body Containing Elastic Parts via Liapunov's Direct Method

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A general and rigorous method for the stability analysis of a spinning body which is part rigid and part elastic is presented. The motion of the body is described by a "hybrid" system of equations, namely a system consisting of both ordinary and partial differential equations. The stability analysis is based on the Liapunov direct method and it works directly with the hybrid set of equations in contrast with the common practice of replacing the partial differential equations with ordinary differential equations by means of modal truncation or spatial discretization. This new approach permits a more rigorous analysis which not only produces sharper stability criteria but also avoids questions as to the effect of the truncation and discretization processes on the results. The general formulation can be used to test the stability of orbiting satellites with various types of elastic members and should also be applicable to the stability analysis of hybrid systems encountered in other areas.

Introduction

THE dynamics problems associated with spinning orbiting bodies have been with us for many years. Problems involving the libration of celestial bodies have stimulated the interest of mathematicians such as Lagrange[1] as early as the mid-eighteenth century. More recently, the advent of artificial satellites has led to renewed interest in these problems. In particular, the problem of attitude stability of spinning spacecraft has received considerable attention.

In a large number of investigations concerned with the attitude stability, the spacecraft is envisioned as a rigid body whose dimensions, although finite, are small compared with the distance to the center of force. This mathematical model permits the assumption that the attitude motion has no effect upon the orbital motion, thus reducing the complexity of the problem by regarding the orbital motion as known. However, in general spacecraft are not entirely rigid and the question remains as to what extent the rigid-body idealization can be justified. A number of investigations concerned with the dynamics of satellites containing elastic parts have been conducted. In the following pages we review some of these studies as a way of introducing the present problem.

In an attempt to explain the tumbling motion of the Explorer I satellite, Thomson and Reiter[2] and Meirovitch[3] have investigated the effect of energy dissipation resulting from the vibration of certain elastic parts of the satellite. On the basis of energy considerations, these investigations concluded that for spin stabilization, spinning motion must be imparted to the satellite about the axis of maximum moment of inertia. Later works by Auelmann,[4] Pringle,[5] and Likins[6] established the usefulness of the Liapunov direct method for the investigation of the attitude stability of satellites, at least for the case of rigid satellites. Subsequently, Pringle[7] used the Liapunov direct method to investigate the stability of a body with connected moving parts. The formulation of Ref. 7, however, is based entirely on ordinary differential equations and is suitable for investigating discrete systems but not distributed ones.

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The motion of spinning bodies containing distributed elastic members is described by sets of both ordinary and partial differential equations. We refer to such sets of differential equations as "hybrid." More pertinent to the present subject is the work by Meirovitch and Nelson[8] who investigated the stability of motion of a satellite containing elastic parts by means of an infinitesimal analysis. Reference 8 represents one of the first attempts to treat rigorously distributed elastic members. The displacement of the elastic members is represented as a series of normal modes multiplying time-dependent generalized coordinates and the effect of truncating the series on the system stability is explored. Also related to the present problem is the one of a satellite with elastically connected moving parts investigated by Nelson and Meirovitch[9] via the Liapunov direct method. In this work the distributed elastic members are simulated by means of discrete masses. The dynamics of a spacecraft consisting of two rigid bodies joined by an elastic structure has been investigated by Robe and Kane.[10] Ignoring gravitational terms, an infinitesimal analysis is carried out for small motions about the simple-spin equilibrium position. The dynamics of satellites containing elastic parts has been further studied by Likins and Wirsching.[11] This latter work considers a discrete system and employs the normal modes to represent elastic displacements.

The Liapunov direct method has been widely used to analyze the stability of discrete systems. In recent years, however, work has been done on extending the Liapunov method to distributed-parameter systems. In this regard we single out the works by Wang[12] and by Parks[13] who applied the method to analyze the stability of partial differential equations associated with elastic and aeroelastic systems.

The present investigation extends the Liapunov direct method to the stability analysis of hybrid systems of differential equations, such as the ones describing the motion of spinning bodies which are part rigid and part elastic. First a variational principle is used to derive the equations of motion. Whereas the rotational motion of the body about its mass center is described by ordinary differential equations, the motion of the elastic members is defined by boundary-value problems consisting of partial differential equations to be satisfied over the entire elastic domain and of certain conditions to be satisfied at the boundaries of this domain. The Liapunov direct method is extended so as to permit working directly with the hybrid set of differential equations, in contrast with the common practice of replacing the partial differential equations with sets of ordinary differential equa-

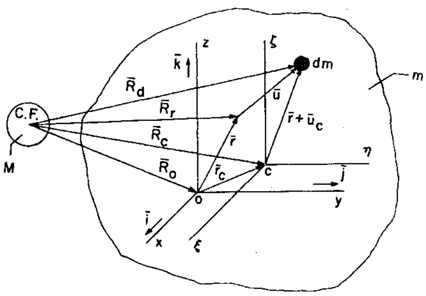

Fig. 1 Flexible body in a central-force field.

tions by means of modal truncation or spatial discretization. These latter two approaches are the ones used in Refs. 8 and 9, respectively. The method presented here makes use of a certain property of Rayleigh's quotient and it does not require the solution of the eigenvalue problem associated with the elastic vibration but only the first eigenvalue. This is particularly important in cases in which the nature of the elastic system does not permit a solution of the eigenvalue problem. The task of obtaining the first eigenvalue in such cases can be carried out either by an approximate method or experimentally. The present approach makes use of the operator notation to formulate the boundary-value problem for a general elastic system, thus permitting the treatment of a large variety of systems. The method should represent a substantial step in the direction of setting the stability analysis of the systems in question on a sound mathematical foundation. The general formulations presented should also be applicable to the stability analysis of hybrid systems encountered in other areas. As an illustration of the method, the case of a gravity-gradient stabilized satellite with flexible antennas is solved.

Differential Equations of Motion

Let us consider a body of total mass m, as shown in Fig. 1. The body, which is capable of elastic deformations, moves in a central-force gravitational field produced by a spherically symmetric body of mass M whose center is denoted by C.F. We shall assume that M is much larger than m so that the motion reduces to the one of m about C.F. Moreover, we assume that the acceleration of M is negligible so that, for all practical purposes, C.F. can be regarded as fixed in an inertial space. The center of mass of m when in the undeformed state is denoted by 0 and the one in the deformed state by c. Next we define a set of body axes xyz as the principal axes of the body in the undeformed configuration and another set of body axes $\xi\eta\zeta$ parallel to axes xyz but with the origin at c instead of 0 (see Fig. 1). The set xyz provides a reference frame for measuring deformations whereas the set $\xi\eta\zeta$ is more convenient for describing the over-all motion. We shall denote by $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the position vector relative to axes xyz of an element of mass dm in the undeformed state and by $\mathbf{u} = u(x,y,z,t)\mathbf{i} + v(x,y,z,t)\mathbf{j} + w(x,y,z,t)\mathbf{k}$ the elastic displacement vector of the element, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along axes x, y, and z, respectively. The components of $\mathbf{u}$ are assumed to be infinitesimally small. Perhaps it should be pointed out here that x, y, and z are spatial coordinates identifying the reference position of an element of mass and not generalized coordinates. They take the place of an index identifying a mass particle in a discrete system and, as such, should be regarded merely as a continuous index which does not depend on time. Hence, the absolute position of dm in the rigid and deformed states is given by the vectors $\mathbf{R}_r = \mathbf{R}_0 + \mathbf{r}$ and $\mathbf{R}_d = \mathbf{R}_0 + \mathbf{r} + \mathbf{u}$, respectively, where $\mathbf{R}_0$ represents the vector from C.F. to 0. Moreover, the absolute position of the body mass center in the deformed state is $\mathbf{R}_c = \mathbf{R}_0 + \mathbf{r}_c$, where $\mathbf{r}_c = x_c\mathbf{i} + y_c\mathbf{j} + z_c\mathbf{k}$ denotes the position of the mass center with respect to axes xyz. The vector $\mathbf{r}_c$ has the value $\mathbf{r}_c = (1/m)\int_m (\mathbf{r} + \mathbf{u})dm = (1/m)\int_m \mathbf{u}\, dm$, since the integral $\int_m \mathbf{r}\, dm$ reduces to zero by virtue of the fact that point 0 represents the body mass center in the undeformed state. Note that all integrations are extended over the undeformed state which serves as the reference state.

In view of the above definitions, the kinetic energy can be shown to have the form

$$T = \tfrac{1}{2}\int_m \dot{\mathbf{R}}_d\cdot\dot{\mathbf{R}}_d\, dm = \tfrac{1}{2}m\dot{\mathbf{R}}_c\cdot\dot{\mathbf{R}}_c + \tfrac{1}{2}\int_m (\dot{\mathbf{r}} + \dot{\mathbf{u}}_c)\cdot(\dot{\mathbf{r}} + \dot{\mathbf{u}}_c)dm \qquad (1)$$

where $\mathbf{u}_c = \mathbf{u} - \mathbf{r}_c = u_c\mathbf{i} + v_c\mathbf{j} + w_c\mathbf{k}$ represents the elastic displacement vector of dm measured with respect to axes $\xi\eta\zeta$. Assuming that axes xyz, hence also axes $\xi\eta\zeta$, rotate with angular velocity $\boldsymbol{\omega}$ relative to an inertial space we have $\dot{\mathbf{r}} + \dot{\mathbf{u}}_c = \dot{\mathbf{u}}_c' + \boldsymbol{\omega}\mathbf{x}(\mathbf{r} + \mathbf{u}_c)$, where $\dot{\mathbf{u}}_c'$ denotes the velocity of dm relative to $\xi\eta\zeta$ due to the elastic effect. It follows that the kinetic energy can be written as

$$T = \tfrac{1}{2}m\dot{\mathbf{R}}_c\cdot\dot{\mathbf{R}}_c + \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J}_d\cdot\boldsymbol{\omega} + [\boldsymbol{\omega}\mathbf{x}\int_m (\mathbf{r} + \mathbf{u}_c)]\cdot\dot{\mathbf{u}}_c'dm + \tfrac{1}{2}\int_m \dot{\mathbf{u}}_c'\cdot\dot{\mathbf{u}}_c'dm \qquad (2)$$

in which $\mathbf{J}_d$ represents the inertia dyadic of the deformed body in terms of components about axes $\xi\eta\zeta$. These components have the expressions

$$\begin{aligned} J_{\xi\xi} &= \int_m[(y + v_c)^2 + (z + w_c)^2]dm \\ J_{\eta\eta} &= \int_m [(x + u_c)^2 + (z + w_c)^2]dm \\ J_{\zeta\zeta} &= \int_m [(x + u_c)^2 + (y + v_c)^2]dm \\ J_{\xi\eta} &= J_{\eta\xi} = \int_m (x + u_c)(y + v_c)dm \\ J_{\xi\zeta} &= J_{\zeta\xi} = \int_m (x + u_c)(z + w_c)dm \\ J_{\eta\zeta} &= J_{\zeta\eta} = \int_m (y + v_c)(z + w_c)dm \end{aligned} \qquad (3)$$

The potential energy arises from two sources, namely gravitational and elastic, where the latter is sometimes referred to as strain energy. The gravitational potential energy has the general form

$$V_G = -K\int_m dm/R_d \qquad (4)$$

where $K = GM$, in which G is the gravitational constant. The reciprocal of the radial distance R_d can be written in the form of the power series

$$R_d^{-1} = [(\mathbf{R}_c + \mathbf{r} + \mathbf{u}_c)\cdot(\mathbf{R}_c + \mathbf{r} + \mathbf{u}_c)]^{-1/2} = R_c^{-1}\{1 - (1/R_c^2)\mathbf{R}_c\cdot(\mathbf{r} + \mathbf{u}_c) - (1/2R_c^2)(\mathbf{r} + \mathbf{u}_c)\cdot(\mathbf{r} + \mathbf{u}_c) + \tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})[2\mathbf{R}_c\cdot(\mathbf{r} + \mathbf{u}_c)/R_c^2]^2 \ldots\} \qquad (5)$$

Introducing Eq. (5) into (4) and integrating over the entire body, we obtain the following expression:

$$V_G = -(Km/R_c) - (K/2R_c^3)[(1 - 3l_\xi^2)J_{\xi\xi} + (1 - 3l_\eta^2)J_{\eta\eta} + (1 - 3l_\zeta^2)J_{\zeta\zeta} + 6l_\xi l_\eta J_{\xi\eta} + 6l_\xi l_\zeta J_{\xi\zeta} + 6l_\eta l_\zeta J_{\eta\zeta}] \qquad (6)$$

where l_ξ, l_η, l_ζ, are the direction cosines between $\mathbf{R}_c$ and axes ξ, η, ζ, respectively. We note that higher-order inertia integrals have beeen ignored in Eq. (6) since in practical cases they are very small.

The elastic potential energy, denoted by V_{EL}, depends on the nature of the elastic members and is in general a function of the partial derivatives of the elastic displacements u, v, w with respect to the spatial variables x, y, z. Since u, v, w differ from u_c, v_c, w_c by the quantities x_c, y_c, z_c, respectively, where the latter do not depend on the spatial variables, V_{EL} can be conveniently expressed in terms of the coordinates u_c, v_c, w_c. To preserve the generality of the formulation, we shall not specify the particular form of V_{EL} at this point.

We are interested in producing a set of differential equations describing the motion of the system; this can be accomplished by means of Hamilton's principle. To this end, we express the body rotational motion in terms of three angular displacements $\theta_i (i = 1,2,3)$ and their associated rates of change. As indicated in the Introduction, the orbital motion may be considered as known by virtue of our assumption that it is not affected by the attitude motion. In fact, we shall be concerned with the case in which the body mass center describes a circular orbit. This leaves us with three coordinates defining the attitude motion and three coordinates describing the components of the elastic displacement at any point x, y, z of the elastic domain D_e. Assuming that the body is part elastic and part rigid, D_e is a subdomain of the domain D, where the latter is a three-dimensional domain corresponding to the entire body.

To define the orientation of the body in space, it will prove convenient to introduce an auxiliary set of axes abc (see Fig. 2), known as an orbital reference frame. This frame is such that axis a coincides with the direction of the vector $\mathbf{R}_c$, axis b is tangent to the orbit and in the direction of the motion, and axis c is normal to the orbit plane. (To avoid any confusion which may arise from the fact that the same symbol c is used to denote the normal axis and the body mass center, explicit reference will be made in the few cases in which axis c is mentioned.) System $\xi\eta\zeta$ is obtained from system abc by means of three rotations θ_1, θ_2, θ_3. Hence, the angular velocity $\boldsymbol{\omega}$ depends on the angular coordinates θ_i and angular velocities $\dot{\theta}_i (i = 1,2,3)$ as well as the orbital angular velocity Ω, where the latter is constant for a circular orbit. Moreover, the direction cosines l_ξ, l_η, l_ζ between a (or $\mathbf{R}_c$) and axes ξ, η, ζ, respectively, depend on the coordinates θ_i. In view of statements made earlier, we assume at this point that V_{EL} is a function of the partial derivatives $\partial^2 u_c/\partial x^2$, $\partial^2 u_c/\partial x \partial y$, . . . , $\partial^2 w_c/\partial z^2$. The assumption that the derivatives are of second order should not be regarded as a limitation on the general formulation. Indeed, the assumption will prove to be an inconsequential step in the process of reaching certain general results. Hence, the system Lagrangian can be written in the general functional form

$$L = T - V_G - V_{EL} = \int_D \hat{L}\left(\theta_i, \dot{\theta}_i, u_c, v_c, \ldots, \dot{w}_c, \frac{\partial^2 u_c}{\partial x^2}, \frac{\partial^2 u_c}{\partial x \partial y} \ldots, \frac{\partial^2 w_c}{\partial z^2}\right) dD \quad (7)$$

where $\hat{L}$ is the Lagrangian density.

For a holonomic system Hamilton's principle has the expression

$$\delta \int_{t_1}^{t_2} L dt = 0 \quad (8)$$

subject to the end conditions

$$\delta\theta_1 = \delta\theta_2 = \delta\theta_3 = \delta u_c = \delta v_c = \delta w_c = 0 \text{ at } t = t_1, t_2 \quad (9)$$

From Eq. (7) we can write the variation of L as follows:

$$\delta L = \int_D \left[\sum_{i=1}^{3}\left(\frac{\partial \hat{L}}{\partial \theta_i}\delta\theta_i + \frac{\partial \hat{L}}{\partial \dot{\theta}_i}\delta\dot{\theta}_i\right) + \frac{\partial \hat{L}}{\partial u_c}\delta u_c + \frac{\partial \hat{L}}{\partial v_c}\delta v_c + \ldots + \frac{\partial \hat{L}}{\partial \dot{w}_c}\delta \dot{w}_c + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x^2)}\delta\left(\frac{\partial^2 u_c}{\partial x^2}\right) + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x \partial y)}\delta\left(\frac{\partial^2 u_c}{\partial x \partial y}\right) + \ldots + \frac{\partial \hat{L}}{\partial(\partial^2 w_c/\partial z^2)}\delta\left(\frac{\partial^2 w_c}{\partial z^2}\right)\right] dD \quad (10)$$

Assuming that the functions u_c, v_c, w_c are well-behaved, we can interchange the variation and differentiation processes so that, after a series of integrations by parts with respect to the spatial variables, we arrive at

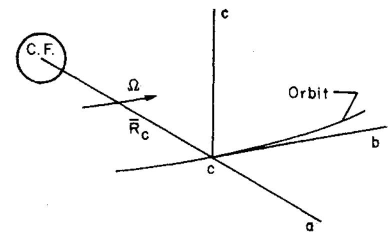

Fig. 2 Orbital axes.

$$\int_D \left[\frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x^2)}\delta\left(\frac{\partial^2 u_c}{\partial x^2}\right) + \frac{\partial \hat{L}}{\partial(\partial^2 u_c/\partial x \partial y)} \times \delta\left(\frac{\partial^2 u_c}{\partial x \partial y}\right) + \ldots + \frac{\partial \hat{L}}{\partial(\partial^2 w_c/\partial z^2)}\left(\frac{\partial^2 w_c}{\partial z^2}\right)\right] dD = \int_{D_e} \mathfrak{L}[u_c, v_c, w_c] \cdot \delta \mathbf{u}_c dD_e + \mathbf{B}_j[u_c, v_c, w_c] \cdot \mathbf{B}_k[u_c, v_c, w_c]|_S, j = 1,2; \quad k = 3,4 \quad (11)$$

where $\mathfrak{L}(\mathfrak{L}_{uc}, \mathfrak{L}_{vc}, \mathfrak{L}_{wc})$ is a differential operator vector with components $\mathfrak{L}_{uc}, \mathfrak{L}_{vc}, \mathfrak{L}_{wc}$ defined over the domain D_e and $\mathbf{B}_j(B_{juc}, B_{jvc}, B_{jwc})$, $\mathbf{B}_k(B_{kuc}, B_{kvc}, B_{kwc})$ are differential operator vectors defined at the surface S bounding the domain D_e, where the latter is recalled as being the domain within which the body possesses elasticity. We note, in passing, that the components of $\mathfrak{L}$ are of order four in our case and the ones of $\mathbf{B}_j$ and $\mathbf{B}_k$ are of order three or less. Introducing Eqs. (10) and (11) into (8), integrating by parts with respect to time, and considering conditions (9), we obtain the ordinary differential equations for the attitude motion

$$(\partial L/\partial \theta_i) - (d/dt)(\partial L/\partial \dot{\theta}_i) = 0, \; i = 1,2,3 \quad (12)$$

and the partial differential equations for the elastic motion

$$\begin{aligned} \frac{\partial \hat{L}}{\partial u_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}_c}\right) + \mathfrak{L}_{uc}[u_c, v_c, w_c] &= 0 \\ \frac{\partial \hat{L}}{\partial v_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{v}_c}\right) + \mathfrak{L}_{vc}[u_c, v_c, w_c] &= 0 \\ \frac{\partial \hat{L}}{\partial w_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}_c}\right) + \mathfrak{L}_{wc}[u_c, v_c, w_c] &= 0 \end{aligned} \quad (13)$$

where Eqs. (13) must be satisfied within the domain D_e. Moreover, the solutions of these equations must satisfy the boundary conditions

$$\mathbf{B}_j[u_c, v_c, w_c] \cdot \mathbf{B}_k[u_c, v_c, w_c] = 0 \text{ on } S, j = 1,2; \quad k = 3,4 \quad (14)$$

We note that the motion of the system is described by a "hybrid" set of equations since Eqs. (12) are ordinary differential equations and Eqs. (13) are partial differential equations.

In any system in which elastic deformations take place there is certain damping present. We shall assume that the damping is internal and independent of the rotational motion of the body. We shall denote the components of the distributed damping forces by $\hat{Q}_{uc}$, $\hat{Q}_{vc}$, $\hat{Q}_{wc}$ so that, whereas Eqs. (12) retain their form, Eqs. (13) become

$$\begin{aligned} \frac{\partial \hat{L}}{\partial u_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{u}_c}\right) + \mathfrak{L}_{uc}[u_c, v_c, w_c] + \hat{Q}_{uc} &= 0 \\ \frac{\partial \hat{L}}{\partial v_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{v}_c}\right) + \mathfrak{L}_{vc}[u_c, v_c, w_c] + \hat{Q}_{vc} &= 0 \\ \frac{\partial \hat{L}}{\partial w_c} - \frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}_c}\right) + \mathfrak{L}_{wc}[u_c, v_c, w_c] + \hat{Q}_{wc} &= 0 \end{aligned} \quad (15)$$

The boundary conditions are not affected by damping so that they remain in the form Eq. (14).

Hamilton's Canonical Equations

We shall find it more convenient to work with a set of first-order Hamiltonian equations instead of the second-order Lagrangian equations. The order here relates to time and not spatial variables. To obtain the set of first-order differential equations, we consider the Hamiltonian defined by

$$H = \sum_{i=1}^{3} \frac{\partial \hat{L}}{\partial \dot{\theta}_i} \dot{\theta}_i + \int_{D_e} \left(\frac{\partial \hat{L}}{\partial \dot{u}_c} \dot{u}_c + \frac{\partial \hat{L}}{\partial \dot{v}_c} \dot{v}_c + \frac{\partial \hat{L}}{\partial \dot{w}_c} \dot{w}_c \right) dD_e - L \quad (16)$$

and note that the Hamiltonian has a "hybrid" form, as it is both a function and a functional at the same time. Introducing the momenta

$$p_{\theta_i} = \partial L/\partial \dot{\theta}_i, \; i = 1,2,3$$
$$\hat{p}_{uc} = \partial \hat{L}/\partial \dot{u}_c, \; \hat{p}_{vc} = \partial \hat{L}/\partial \dot{v}_c, \; \hat{p}_{wc} = \partial \hat{L}/\partial \dot{w}_c \quad (17)$$

where the latter three are momentum densities, the Hamiltonian assumes the form

$$H = \sum_{i=1}^{3} p_{\theta_i} \dot{\theta}_i + \int_{D_e} (\hat{p}_{uc}\dot{u}_c + \hat{p}_{vc}\dot{v}_c + \hat{p}_{wc}\dot{w}_c) dD_e - L$$
$$= \int_{D_e} \hat{H} \left(\theta_i, u_c, v_c, w_c, p_{\theta_i}, \hat{p}_{uc}, \hat{p}_{vc}, \hat{p}_{wc}, \frac{\partial^2 u_c}{\partial x^2}, \frac{\partial^2 u_c}{\partial x \partial y}, \ldots, \frac{\partial^2 w_c}{\partial z^2} \right) dD_e \quad (18)$$

in which $\hat{H}$ is the Hamiltonian density. Considering both forms of H in (18), we can write the variation of the Hamiltonian as follows:

$$\delta H = \sum_{i=1}^{3} (\delta p_{\theta_i} \dot{\theta}_i + p_{\theta_i} \delta \dot{\theta}_i) + \int_{D_e} (\delta \hat{p}_{uc} \dot{u}_c + \hat{p}_{uc} \delta \dot{u}_c + \ldots + \hat{p}_{wc} \delta \dot{w}_c) dD_e - \sum_{i=1}^{3} \left(\frac{\partial L}{\partial \theta_i} \delta \theta_i + \frac{\partial L}{\partial \dot{\theta}_i} \delta \dot{\theta}_i \right) - \int_{D_e} \left[\frac{\partial \hat{L}}{\partial u_c} \delta u_c + \frac{\partial \hat{L}}{\partial v_c} \delta v_c + \ldots + \frac{\partial \hat{L}}{\partial \dot{w}_c} \delta \dot{w}_c + \frac{\partial \hat{L}}{\partial (\partial^2 u_c/\partial x^2)} \delta \left(\frac{\partial^2 u_c}{\partial x^2} \right) + \frac{\partial \hat{L}}{\partial (\partial^2 u_c/\partial x \partial y)} \delta \left(\frac{\partial^2 u_c}{\partial x \partial y} \right) + \ldots + \frac{\partial \hat{L}}{\partial (\partial^2 w_c/\partial z^2)} \times \delta \left(\frac{\partial^2 w_c}{\partial z^2} \right) \right] dD_e = \sum_{i=1}^{3} \left(\frac{\partial H}{\partial \theta_i} \delta \theta_i + \frac{\partial H}{\partial p_{\theta_i}} \delta p_{\theta_i} \right) + \int_{D_e} \left[\frac{\partial \hat{H}}{\partial u_c} \delta u_c + \frac{\partial \hat{H}}{\partial v_c} \delta v_c + \ldots + \frac{\partial \hat{H}}{\partial \hat{p}_{wc}} \delta \hat{p}_{wc} + \frac{\partial \hat{H}}{\partial (\partial^2 u_c/\partial x^2)} \delta \left(\frac{\partial^2 u_c}{\partial x^2} \right) + \frac{\partial \hat{H}}{\partial (\partial^2 u_c/\partial x \partial y)} \delta \left(\frac{\partial^2 u_c}{\partial x \partial y} \right) + \ldots + \frac{\partial \hat{H}}{\partial (\partial^2 w_c/\partial z^2)} \delta \left(\frac{\partial^2 w_c}{\partial z^2} \right) \right] dD_e \quad (19)$$

Recalling definitions (17) and comparing coefficients of like variations in both forms of (19), we obtain the Hamiltonian equations

$$\dot{\theta}_i = \partial H/\partial p_{\theta_i}, \; i = 1,2,3$$
$$\dot{u}_c = \partial \hat{H}/\partial \hat{p}_{uc}, \; \dot{v}_c = \partial \hat{H}/\partial \hat{p}_{vc}, \; \dot{w}_c = \partial \hat{H}/\partial \hat{p}_{wc} \text{ at every point of } D_e$$
$$\dot{p}_{\theta_i} = -\partial H/\partial \theta_i, \; i = 1,2,3$$
$$\left.\begin{aligned} \dot{\hat{p}}_{uc} &= -(\partial \hat{H}/\partial u_c) + \mathcal{L}_{uc}[u_c,v_c,w_c] + \hat{Q}_{uc} \\ \dot{\hat{p}}_{vc} &= -(\partial \hat{H}/\partial v_c) + \mathcal{L}_{vc}[u_c,v_c,w_c] + \hat{Q}_{vc} \\ \dot{\hat{p}}_{wc} &= -(\partial \hat{H}/\partial w_c) + \mathcal{L}_{wc}[u_c,v_c,w_c] + \hat{Q}_{wc} \end{aligned}\right\} \text{at every point of } D_e \quad (20)$$

Note that to obtain the second half of Eqs. (20) use has been made of Lagrange's equations, Eqs. (12) and (15). Of course, the boundary conditions, Eqs. (14), remain the same.

Stability of Motion of a Dynamical System

Let us consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}) \quad (21)$$

For a discrete system $\mathbf{x} = \mathbf{x}(t)$ represents a vector in a finite dimensional vector space whereas for a distributed system $\mathbf{x} = \mathbf{x}(P,t)$ represents an element of a function space in which P denotes a point with the spatial coordinates x,y,z in the domain D. We shall refer to a system which is partly discrete and partly distributed as a hybrid system. The state of a hybrid system at time t is given by an element in a space S which can be regarded as the cartesian product of the finite dimensional vector space and the function space. The motion of the system can be represented as a path in that space. If Eq. (21) represents a set of canonical equations, then the motion of the dynamical system can be regarded as a succession of infinitesimal contact transformations possessing the group-property. The properties characterizing the group are as follows: 1) the identity transformation belongs to this class, 2) two successive transformations are commutative and the result is also a contact transformation, 3) two contact transformations satisfy the associative law, and 4) the inverse of a contact transformation is also a contact transformation. Hence, the motion of the system may be interpreted as a continuous mapping of the space S onto itself. For canonical systems of equations half of the elements of $\mathbf{x}$ represent generalized coordinates and the remaining half represent the conjugate momenta. Moreover, the space S is simply the phase space.

A solution of Eq. (21) satisfying

$$\mathbf{X}(\mathbf{x}) = \mathbf{0} \quad (22)$$

represents a singular point or an equilibrium position. We shall be interested in the stability of the solutions in the neighborhood of equilibrium positions. Without loss of generality, we can assume that the equilibrium point coincides with the origin so that we shall be concerned with the equilibrium of the trivial solution. Denoting the integral curve at a given time $t_0 > 0$ by $\mathbf{x}(t_0) = \mathbf{x}_0$, and assuming that the origin is an isolated singularity, we can introduce the following definitions due to Liapunov:

1) The null solution is stable in the sense of Liapunov if for any arbitrary positive ϵ and time t_0 there exists a $\delta(\epsilon,t_0) > 0$ such that if the inequality

$$||\mathbf{x}_0|| < \delta \quad (23)$$

is satisfied, then the inequality

$$||\mathbf{x}(t)|| < \epsilon, \; t_0 \le t < \infty \quad (24)$$

is implied. If δ is independent of t_0 the stability is said to be uniform.

2) The null solution is asymptotically stable if it is Liapunov stable and in addition

$$\lim_{t \to \infty} ||\mathbf{x}(t)|| = 0 \quad (25)$$

Similarly, if Eq. (25) holds, then a uniformly stable solution is said to be uniformly asymptotically stable. For autonomous systems stability is always uniform.

3) The null solution is said to be unstable if for any arbitrarily small ϵ and any time t_0 such that

$$\|\mathbf{x}_0\| < \epsilon \tag{26}$$

we have at some other finite time t_1 the situation

$$\|\mathbf{x}(t_1)\| = \epsilon,\ t_1 > t_0 \tag{27}$$

To test the stability of the trivial solution, we shall use Liapunov's direct method which is based on the differential equation (21) but does not require the solution of this equation. To introduce the concepts, we confine ourselves to autonomous systems and consider a scalar function $U(\mathbf{x})$ such that $U(\mathbf{0}) = 0$. The total time derivative of U along a trajectory of system (21) is defined by

$$\dot{U} = dU/dt = \nabla U \cdot \dot{\mathbf{x}} = \nabla U \cdot \mathbf{X} \tag{28}$$

where ∇U is the gradient of the scalar function U. In the case of a hybrid system U is both a function and a functional at the same time, as the dependent variables corresponding to the distributed portion of the system appear in U in integrated form.

Next we consider the following theorems:

Theorem 1—If there exists for the system (21) a positive (negative) definite function $U(\mathbf{x})$ whose total time derivative $\dot{U}(\mathbf{x})$ is negative (positive) semidefinite along every trajectory of Eq. (21), then the trivial solution $\mathbf{x} = \mathbf{0}$ is stable.

Theorem 2—If the conditions of Theorem 1 are satisfied and if in addition the set of points at which $\dot{U}(\mathbf{x})$ is zero contains no nontrivial positive half-trajectory $\mathbf{x}(t)$, $t \geq t_0$, then the trivial solution is asymptotically stable.

Theorem 3—If there exists for the system (21) a function $U(\mathbf{x})$ whose total time derivative $\dot{U}(\mathbf{x})$ is positive (negative) definite along every trajectory of Eq. (21) and the function itself can assume positive (negative) values in the neighborhood of the origin, then the trivial solution is unstable.

Theorem 4—Suppose that a function $U(\mathbf{x})$ such as in Theorem 3 exists but for which $\dot{U}(\mathbf{x})$ is only positive (negative) semidefinite and, in addition, the set of points at which $\dot{U}(\mathbf{x})$ is zero contains no nontrivial positive half-trajectory $\mathbf{x}(t)$, $t \geq t_0$. Suppose further that in every neighborhood of the origin there is a point $\mathbf{x}(t_0) = \mathbf{x}_0$ such that for arbitrary $t_0 \geq 0$ we have $U(\mathbf{x}_0) > 0 (<0)$. Then the trivial solution is unstable and the trajectories $\mathbf{x}(\mathbf{x}_0,t_0,t)$ for which $U(\mathbf{x}_0) > 0$ (<0) must leave the open domain $\|\mathbf{x}\| < \epsilon$ as the time t increases.

A function U satisfying any of the preceding theorems is referred to as a Liapunov function. Theorems 1 and 3 are due to Liapunov, whereas Theorems 2 and 4 are due to Krasovskii. A more detailed discussion of the theorems can be found in the text by L. Meirovitch (see Ref. 14, Sec. 6.7).

The Hamiltonian as a Liapunov Function

We shall show next that under certain circumstances the Hamiltonian can be used as a Liapunov function. Taking the total time derivative of H from the first form of Eq. (18) and using Eqs. (12) and (15), as well as boundary conditions (14) and definitions (17), we obtain

$$\dot{H} = \int_{D_e} (\hat{Q}_{uc}\dot{u}_c + \hat{Q}_{vc}\dot{v}_c + \hat{Q}_{wc}\dot{w}_c)dD_e \tag{29}$$

Next we assume that the damping forces are such that $\dot{H}$ is negative semidefinite

$$\dot{H} \leq 0 \tag{30}$$

Moreover, due to coupling, the forces $\hat{Q}_{uc}$, $\hat{Q}_{vc}$, $\hat{Q}_{wc}$ are never identically zero at every point of the phase space but they reduce to zero at an equilibrium point. Hence, if the Hamiltonian H is positive definite at an equilibrium point, then by Theorem 2, H can be regarded as a Liapunov function and the equilibrium point under consideration as asymptotically stable. On the other hand, if H is not positive definite and there are points for which it is negative, then by Theorem 4 the equilibrium point is unstable.

In the event there are ignorable coordinates in the system, so that certain first integrals exist, then a combination of the Hamiltonian and the first integrals in question may prove to be a suitable Liapunov function.

In view of the preceding discussion, we shall consider the Hamiltonian as a Liapunov function. As indicated by Eq. (22), the equilibrium positions are those rendering the right sides of Eqs. (20) equal to zero. Hence, the equilibrium positions are the solutions of the equations

$$\partial H/\partial p_{\theta_i} = 0,\ -\partial H/\partial \theta_i = 0,\ i = 1,2,3$$

$$\left.\begin{aligned} \frac{\partial \hat{H}}{\partial \hat{p}_{uc}} = \frac{\partial \hat{H}}{\partial \hat{p}_{vc}} = \frac{\partial \hat{H}}{\partial \hat{p}_{wc}} = 0 \\ -\frac{\partial \hat{H}}{\partial u_c} + \mathfrak{L}_{uc}[u_c,v_c,w_c] = 0 \\ -\frac{\partial \hat{H}}{\partial v_c} + \mathfrak{L}_{vc}[u_c,v_c,w_c] = 0 \\ -\frac{\partial \hat{H}}{\partial w_c} + \mathfrak{L}_{wc}[u_c,v_c,w_c] = 0 \end{aligned}\right\} \text{at every point of } D_e \tag{31}$$

In the general case the kinetic energy has the form

$$T = T_2 + T_1 + T_0 \tag{32}$$

where T_2 is quadratic and T_1 is linear in the generalized velocities whereas T_0 is a function of the generalized coordinates alone. In this case, the Hamiltonian can be expressed in the form

$$H = T_2 - T_0 + V_G + V_{EL} \tag{33}$$

The elastic potential energy requires further elaboration. We shall confine ourselves to the case in which the elastic displacements u, v, and w are independent of one another, in which case V_{EL} can be shown to reduce to

$$V_{EL} = \tfrac{1}{2}\int_{D_e} (u\mathfrak{L}_u[u] + v\mathfrak{L}_v[v] + w\mathfrak{L}_w[w])dD_e \tag{34}$$

where u, v, and w are subject to the boundary conditions

$$\left.\begin{aligned} B_{ju}[u] = 0 \text{ or } B_{ku}[u] = 0 \\ B_{jv}[v] = 0 \text{ or } B_{kv}[v] = 0 \\ B_{jw}[w] = 0 \text{ or } B_{kw}[w] = 0 \end{aligned}\right\} \text{on } S,\ j = 1,2;\quad k = 3,4 \tag{35}$$

Under these conditions, the eigenvalue problem corresponding to the elastic motion separates into the three individual eigenvalue problems defined by the differential equations

$$\mathfrak{L}_u[u] = \Lambda_u^2\rho u,\ \mathfrak{L}_v[v] = \Lambda_v^2\rho v,\ \mathfrak{L}_w[w] = \Lambda_w^2\rho w \tag{36}$$

which must be satisfied over the domain D_e and by the boundary conditions (35), respectively.

At this point let us define the Rayleigh quotient associated with u as follows:

$$R_u(u) = \int_{D_e} u\mathfrak{L}_u[u]dD_e/\int_{D_e} \rho u^2 dD_e \tag{37}$$

For a positive definite operator $\mathfrak{L}_u$, the quotient $R_u(u)$ is always larger than zero. Moreover, denoting by Λ_{1u}^2 the lowest eigenvalue associated with the vibration u, it can be shown that (see Ref. 15, Sec. 5–14)

$$R_u(u) \geq \Lambda_{1u}^2 \tag{38}$$

Analogous statements can be made with regard to the displacements v and w. It follows from (37) and (38), together with similar expressions for v and w, that

$$\begin{aligned} V_{EL} &= \tfrac{1}{2}\int_{D_e}(u\mathfrak{L}_u[u] + v\mathfrak{L}_v[v] + w\mathfrak{L}_w[w])dD_e \\ &\geq \tfrac{1}{2}\int_{D_e} \rho(\Lambda_{1u}^2u^2 + \Lambda_{1v}^2v^2 + \Lambda_{1w}^2w^2)dD_e \end{aligned} \tag{39}$$

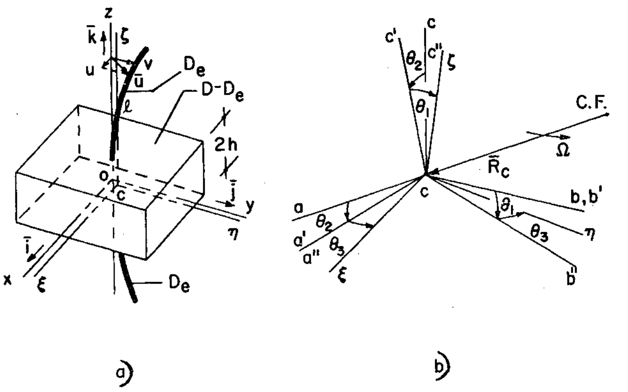

Fig. 3 a) Flexible satellite, b) satellite angular displacements.

But u_c,v_c,w_c differ from u,v,w only by the rigid-body translations x_c,y_c,z_c, respectively. In view of that, it is not difficult to show that Eqs. (34) and (35) remain valid if u,v, and w are correspondingly replaced by u_c,v_c, and w_c. Moreover, it follows that

$$V_{EL} \geq \tfrac{1}{2}\int_{D_e} \rho(\Lambda_{1u}{}^2 u_c{}^2 + \Lambda_{1v}{}^2 v_c{}^2 + \Lambda_{1w}{}^2 w_c{}^2) dD_e \quad (40)$$

Hence, for the asymptotic stability of an equilibrium point it is sufficient to show that

$$\kappa = T_2 - T_0 + V_G + \tfrac{1}{2}\int_{D_e} \rho(\Lambda_{1u}{}^2 u_c{}^2 + \Lambda_{1v}{}^2 v_c{}^2 + \Lambda_{1w}{}^2 w_c{}^2) dD_e > 0 \quad (41)$$

in the neighborhood of the equilibrium point in question. To this end, it may be possible to employ Sylvester's criterion (see Ref. 14, Sec. 6.7) by writing an equivalent Hessian density matrix associated with κ for every point of the domain D_e and proving that the matrix is positive definite at every such point. This, in turn, ensures the satisfaction of inequality (41).

In conclusion, the analysis does not necessitate the knowledge of the modes of vibration but only of the lowest eigenvalues. For complicated systems these eigenvalues can be obtained by approximate methods or experimentally.

The Case of a Rigid Satellite with Flexible Appendages in a Circular Orbit

As an illustration of the general formulation, let us consider a rigid satellite with flexible appendages, such as antennas, where the satellite mass center moves in a circular orbit. The satellite is shown in Fig. 3a, where the principal axes of the body in the undeformed position coincide with axes xyz. Note that in the undeformed state the antennas lie along the z axis. The system shown in Fig. 3a is analogous to the one of Ref. 9 with the significant exception that in this case the antennas are represented by a distributed-parameter model whereas in Ref. 9 they are simulated by discrete masses. As a result, the present formulation is of a hybrid type, in contrast with the one of Ref. 9 which consists entirely of ordinary differential equations.

For a circular orbit we have $R_c = |\mathbf{R}_c| =$ const so that the term involving $\dot{\mathbf{R}}_c$ in the kinetic energy, Eq. (2), can be ignored. The flexible members are assumed to be sufficiently thin that they can be regarded as one-dimensional. Hence, the domain of the elastic continuum is simply $D_e\!: h < z < h + l,\ -(h + l) < z < -h$. It follows that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over $D - D_e$ and $\mathbf{r} = z\mathbf{k}$ over D_e. Moreover, it is assumed that the elastic motion takes place in the x and y directions only so that $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$. It follows that $\mathbf{r}_c = x_c\mathbf{i} + y_c\mathbf{j}$ and $\mathbf{u}_c = u_c\mathbf{i} + v_c\mathbf{j}$. The angular velocity has the general form $\boldsymbol{\omega} = \omega_\xi\mathbf{i} + \omega_\eta\mathbf{j} + \omega_\zeta\mathbf{k}$. Introducing these relations into Eq. (2), and ignoring the constant term, we obtain

$$T = \tfrac{1}{2}(J_{\xi\xi}\omega_\xi{}^2 + J_{\eta\eta}\omega_\xi{}^2 + J_{\zeta\zeta}\omega_\zeta{}^2 - 2J_{\xi\eta}\omega_\xi\omega_\eta - 2J_{\xi\zeta}\omega_\xi\omega_\zeta - 2J_{\eta\zeta}\omega_\eta\omega_\zeta) + \int_{D_e} \rho[z(\omega_\eta\ddot{u}_c - \omega_\xi\dot{v}_c) + \omega_\zeta(u_c\dot{v}_c - \dot{u}_c v_c)]dz + \tfrac{1}{2}\int_{D_e} \rho(\dot{u}_c{}^2 + \dot{v}_c{}^2)dz \quad (42)$$

where ρ denotes the mass per unit length of the flexible rods and

$$J_{\xi\xi} = \int_D \gamma[(y + v_c)^2 + z^2]dD = A + \int_{D_e} \rho v_c{}^2 dz$$
$$J_{\eta\eta} = \int_D \gamma[(x + u_c)^2 + z^2]dD = B + \int_{D_e} \rho u_c{}^2 dz$$
$$J_{\zeta\zeta} = \int_D \gamma[(x + u_c)^2 + (y + v_c)^2]dD = C + \int_{D_e} \rho(u_c{}^2 + v_c{}^2)dz \quad (43)$$
$$J_{\xi\eta} = \int_D \gamma(x + u_c)(y + v_c)dD = \int_{D_e} \rho u_c v_c\, dz$$
$$J_{\xi\zeta} = \int_D \gamma(x + u_c)z\, dD = \int_{D_e} \rho z u_c dz$$
$$J_{\eta\zeta} = \int_D \gamma(y + v_c)z\, dD = \int_{D_e} \rho z v_c dz$$

are the mass moments and products of inertia of the deformed body about axes $\xi\eta\zeta$, in which γ is the mass density and A,B,C are the principal mass moments of the entire body in the undeformed state.

At this point it becomes necessary to produce explicit expressions for ω_ξ, ω_η, ω_ζ in terms of θ_1, θ_2, θ_3, $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$, and Ω, as well as for l_ξ, l_η, l_ζ in terms of θ_1, θ_2, θ_3. Recalling that axes abc rotate relative to an inertial space with the orbital angular velocity Ω about axis c, the angular velocity components can be shown to have the expressions

$$\omega_\xi = -\Omega(s\theta_2\, c\theta_3 + s\theta_1\, c\theta_2\, s\theta_3) - \dot{\theta}_1\, c\theta_3 + \dot{\theta}_2\, c\theta_1\, s\theta_3$$
$$\omega_\eta = \Omega(s\theta_2\, s\theta_3 - s\theta_1\, c\theta_2\, c\theta_3) + \dot{\theta}_1\, s\theta_3 + \dot{\theta}_2\, c\theta_1\, c\theta_3 \quad (44)$$
$$\omega_\zeta = \Omega\, c\theta_1\, c\theta_2 + \dot{\theta}_2\, s\theta_1 + \dot{\theta}_3$$

where $s\theta_1 = \sin\theta_1$, $c\theta_1 = \cos\theta_1$, etc. Since the direction of the radius vector $\mathbf{R}_c$ coincides with axis a at all times, the direction cosines can be written as

$$l_\xi = l_{\xi a} = c\theta_2\, c\theta_3 - s\theta_1\, s\theta_2\, s\theta_3$$
$$l_\eta = l_{\eta a} = -\, c\theta_2\, s\theta_3 - s\theta_1\, s\theta_2\, c\theta_3 \quad (45)$$
$$l_\zeta = l_{\zeta a} = c\theta_1\, s\theta_2$$

Introducing expressions (44) into (42) we obtain the kinetic energy T which can be written in the general form (32). From inequality (41), however, we conclude that T_1 does not enter into the testing function κ. On the other hand, T_2 and T_0 do appear in κ so that further attention will be given only to these two terms. Their expressions are

$$T_2 = \tfrac{1}{2}\{\dot{\theta}_1{}^2(J_{\xi\xi}c^2\theta_3 + J_{\eta\eta}\, s^2\theta_3 + 2J_{\xi\eta}s\theta_3 c\theta_3) + \dot{\theta}_2{}^2(J_{\xi\xi}c^2\theta_1 s^2\theta_3 + J_{\eta\eta}c^2\theta_1 c^2\theta_3 + J_{\zeta\zeta}s^2\theta_1 - 2J_{\xi\eta}c^2\theta_1 s\theta_3 c\theta_3 - 2J_{\xi\zeta}s\theta_1 c\theta_1 s\theta_3 - 2J_{\eta\zeta}s\theta_1 c\theta_1 c\theta_3) + \dot{\theta}_3{}^2 J_{\zeta\zeta} + 2\dot{\theta}_1\dot{\theta}_2[(J_{\eta\eta} - J_{\xi\xi})c\theta_1 s\theta_3 c\theta_3 + J_{\xi\eta}c\theta_1(c^2\theta_3 - s^2\theta_3) + J_{\xi\zeta}s\theta_1 c\theta_3 - J_{\eta\zeta}s\theta_1 s\theta_3] + 2\dot{\theta}_1\dot{\theta}_3(J_{\xi\zeta}\, c\theta_3 - J_{\eta\zeta}s\theta_3) + 2\dot{\theta}_2\dot{\theta}_3(J_{\zeta\zeta}s\theta_1 - J_{\xi\zeta}c\theta_1 s\theta_3 - J_{\eta\zeta}c\theta_1 c\theta_3)\} + (\dot{\theta}_1 c\theta_3 - \dot{\theta}_2 c\theta_1 s\theta_3)\int_{D_e} \rho z\dot{v}_c dz + (\dot{\theta}_1 s\theta_3 + \dot{\theta}_2 c\theta_1 c\theta_3)\int_{D_e} \rho z\dot{u}_c\, dz + (\dot{\theta}_2 s\theta_1 + \dot{\theta}_3)\int_{D_e} \rho(u_c\dot{v}_c - \dot{u}_c v_c)dz + \tfrac{1}{2}\int_{D_e} \rho(\dot{u}_c{}^2 + \dot{v}_c{}^2)dz \quad (46)$$

and

$$T_0 = \tfrac{1}{2}\Omega^2[J_{\xi\xi}(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3)^2 + J_{\eta\eta}(s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3)^2 + J_{\zeta\zeta}c^2\theta_1 c^2\theta_2 + 2J_{\xi\eta}(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3)(s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3) + 2J_{\xi\zeta}c\theta_1 c\theta_2(s\theta_2 c\theta_3 + s\theta_1 c\theta_2 s\theta_3) - 2J_{\eta\zeta}c\theta_1 c\theta_2(s\theta_2 s\theta_3 - s\theta_1 c\theta_2 c\theta_3)] \quad (47)$$

But for a circular orbit we have the relation $\Omega^2 = K/R_c^3$ so that, introducing Eqs. (45) into Eq. (6), and ignoring the constant term, we obtain

$$V_G = \tfrac{1}{2}\Omega^2\{J_{\xi\xi}[3(c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3)^2 - 1] + J_{\eta\eta}[3(c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3)^2 - 1] + J_{\zeta\zeta}(3c^2\theta_1 s^2\theta_2 - 1) + 6J_{\xi\eta}(c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3) \times (c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3) - 6J_{\xi\zeta}(c\theta_2 c\theta_3 - s\theta_1 s\theta_2 s\theta_3)c\theta_1 s\theta_2 + 6J_{\eta\zeta}(c\theta_2 s\theta_3 + s\theta_1 s\theta_2 c\theta_3)c\theta_1 s\theta_2\} \quad (48)$$

Assuming that the cross sectional area moments of inertia associated with axes x and y are I_v and I_u, respectively, the elastic potential energy for flexure in the x and y directions has the expression

$$V_{EL} = \tfrac{1}{2}\int_{D_e} [EI_u(\partial^2 u/\partial z^2)^2 + EI_v(\partial^2 v/\partial z^2)^2]dz \quad (49)$$

Integrating Eq. (49) by parts, we obtain

$$V_{EL} = \frac{1}{2}\left[\frac{\partial u}{\partial z} EI_u \frac{\partial^2 u}{\partial z^2} + \frac{\partial v}{\partial z} EI_v \frac{\partial^2 v}{\partial z^2}\right]\Bigg|_S - \frac{1}{2}\left[u\frac{\partial}{\partial z}\left(EI_u \frac{\partial^2 u}{\partial z^2}\right) + v\frac{\partial}{\partial z}\left(EI_v\frac{\partial^2 v}{\partial z^2}\right)\right]\Bigg|_S + \frac{1}{2}\int_{D_e}\left[u\frac{\partial^2}{\partial z^2}\left(EI_u\frac{\partial^2 u}{\partial z^2}\right) + v\frac{\partial^2}{\partial z^2}\left(EI_v\frac{\partial^2 v}{\partial z^2}\right)\right]dz \quad (50)$$

where S denotes the boundary of the domain D_e, which in our particular case represents the points $z = \pm h, \pm (h + l)$. Hence, the operators $\mathfrak{L}_u$ and $\mathfrak{L}_v$ have the form

$$\mathfrak{L}_u = (\partial^2/\partial z^2)[EI_u(\partial^2/\partial z^2)], \quad \mathfrak{L}_v = \partial^2/\partial z^2[EI_v(\partial^2/\partial z^2)] \quad (51)$$

Moreover, from the expressions to be evaluated on S in Eq. (50), we conclude that the operators associated with the boundary conditions are

$$B_{1u} = EI_u\frac{\partial^2}{\partial z^2}, \quad B_{2u} = \frac{\partial}{\partial z}\left(EI_u\frac{\partial^2}{\partial z^2}\right), \quad B_{3u} = \frac{\partial}{\partial z}, \quad B_{4u} = 1$$
$$B_{1v} = EI_v\frac{\partial^2}{\partial z^2}, \quad B_{2v} = \frac{\partial}{\partial z}\left(EI_v\frac{\partial^2}{\partial z^2}\right), \quad B_{3v} = \frac{\partial}{\partial z}, \quad B_{4v} = 1 \quad (52)$$

Since the rods are fixed at $z = \pm h$ and free at the ends $z = \pm (h + l)$, we have the boundary conditions

$$B_{3u}[u] = B_{4u}[u] = B_{3v}[v] = B_{4v}[v] = 0 \text{ at } z = \pm h$$
$$B_{1u}[u] = B_{2u}[u] = B_{1v}[v] = B_{2v}[v] = 0 \text{ at } z = \pm(h + l) \quad (53)$$

with the implication that

$$V_{EL} = \tfrac{1}{2}\int_{D_e} (u\mathfrak{L}_u[u] + v\mathfrak{L}_v[v])dz \quad (54)$$

Clearly the eigenvalue problems associated with u and v are independent of one another. It is also clear that the results do not change if we replace u and v by u_c and v_c, respectively, in Eq. (49) so that inequality (40) (with $w_c = 0$) holds.

From Eqs. (31) it is not difficult to show that the equilibrium positions of the rotating body must satisfy

$$\dot\theta_i = 0, \; i = 1,2,3 \text{ and } \dot u_c = \dot v_c = 0 \quad (55)$$

as well as

$$-\partial H/\partial\theta_i = 0, \; i = 1,2,3, \; -\partial\hat H/\partial u_c + \mathfrak{L}_{uc}[u_c] = -\partial\hat H/\partial v_c + \mathfrak{L}_{vc}[v_c] = 0 \quad (56)$$

Examining Eqs. (56), it is possible to distinguish three equilibrium configurations, namely E_1: $\theta_1 = \theta_2 = \theta_3 = u_c = v_c = 0$, E_2: $\theta_1 = \theta_3 = u_c = v_c = 0$, $\theta_2 = \pi/2$, and E_3: $\theta_1 = -\pi/2$, $\theta_2 = \theta_3 = u_c = v_c = 0$. We shall concern ourselves with the stability of the equilibrium point E_1. The remaining equilibrium points, E_2 and E_3, can be studied in a similar manner, although for the point E_3 a different set of rotations is advisable as the velocities $\dot\theta_2$ and $\dot\theta_3$ are collinear in this case.

We have indicated in the preceding section that $\dot H$ is negative semidefinite so that the equilibrium point $\theta_i = \dot\theta_i = 0$ $(i = 1,2,3)$, $u_c = v_c = \dot u_c = \dot v_c = 0$ is asymptotically stable if H is positive definite in the neighborhood of that point. But $H \geq \kappa$ so that it is sufficient to show that κ is positive definite in the neighborhood of the equilibrium point in question, where

$$\kappa = T_2 - T_0 + V_G + \tfrac{1}{2}\int_{D_e}\rho(\Lambda_{1u}^2 u_c^2 + \Lambda_{1v}^2 v_c^2)dz \quad (57)$$

n which Λ_{1u}^2 and Λ_{1v}^2 are the lowest eigenvalues corresponding to the flexural vibration of the elastic members in the x and y directions, respectively. Introducing Eqs. (46)–(48), in conjunction with expressions (43), into Eq. (57), retaining only terms through second-order in the variables and their time derivatives, and ignoring constant terms, we arrive at

$$\kappa|_{E_1} = \tfrac{1}{2}(A\dot\theta_1^2 + B\dot\theta_2^2 + C\dot\theta_3^2) + \dot\theta_1\int_{D_e}\rho z\dot v_c dz + \dot\theta_2\int_{D_e}\rho z\dot u_c\, dz + \tfrac{1}{2}\int_{D_e}\rho(\dot u_c^2 + \dot u_c^2)dz + \tfrac{1}{2}\Omega^2[(C - B)\theta_1^2 + 4(C - A)\theta_2^2 + 3(B - A)\theta_3^2 - 2\theta_1\int_{D_e}\rho z v_c dz - 8\theta_2\int_{D_e}\rho z u_c\, dz - 3\int_{D_e}\rho u_c^2 dz] + \tfrac{1}{2}\Lambda_{1u}^2\int_{D_e}\rho u_c^2 dz + \tfrac{1}{2}\Lambda_{1v}^2\int_{D_e}\rho v_c^2 dz \quad (58)$$

To test $\kappa|_{E_1}$ for positive definiteness, we use Sylvester's criterion. To this end, we introduce the notation $\hat A = A/2l$, $\hat B = B/2l$, $\hat C = C/2l$, and write the Hessian density matrix corresponding to any point of the domain D_e

$$[\hat\kappa]_{E1} = \left[\begin{array}{ccccc|ccccc}
\hat A & 0 & 0 & 0 & \rho z & & & & & \\
0 & \hat B & 0 & \rho z & 0 & & & & & \\
0 & 0 & \hat C & 0 & 0 & & & [0] & & \\
0 & \rho z & 0 & \rho & 0 & & & & & \\
\rho z & 0 & 0 & 0 & \rho & & & & & \\
\hline
 & & & & & (\hat C - \hat B)\Omega^2 & 0 & 0 & 0 & -\rho z\Omega^2 \\
 & & & & & 0 & 4(\hat C - \hat A)\Omega^2 & 0 & -4\rho z\Omega^2 & 0 \\
 & & [0] & & & 0 & 0 & 3(\hat B - \hat A)\Omega^2 & 0 & 0 \\
 & & & & & 0 & -4\rho z\Omega^2 & 0 & \rho(\Lambda_{1u}^2 - 3\Omega^2) & 0 \\
 & & & & & -\rho z\Omega^2 & 0 & 0 & 0 & \rho\Lambda_{1v}^2
\end{array}\right] \quad (59)$$

which must be positive definite for any value of z. A simple application of Sylvester's criterion reveals that the Hessian density matrix is positive definite everywhere in the domain D_e if the following conditions are satisfied

$$\hat C > \hat B > \hat A > (\rho z^2)_{\max}$$
$$\Lambda_{1u}^2 > 3\Omega^2 + [4\Omega^2/(\hat C - \hat A)](\rho z^2)_{\max} \quad (60)$$
$$\Lambda_{1v}^2 > [\Omega^2/(\hat C - \hat B)](\rho z^2)_{\max}$$

The satisfaction of inequalities (60) ensures that the Hamiltonian is positive definite in the neighborhood of the equilibrium point E_1 in which case the point E_1 is asymptotically stable. It should be noted that the equilibrium point E_1 represents the configuration corresponding to gravity-gradient stabilization.

For uniform rods of equal moments of inertia, $I_u = I_v$, the above conditions lead to

$$C > B > A > 2m_e(h + l)^2$$
$$\Lambda_1^2 > 3\Omega^2 + 8m_e\Omega^2(h + l)^2/(C - A) \quad (61)$$
$$\Lambda_1^2 > 2m_e\Omega^2(h + l)^2/(C - B)$$

where $\Lambda_1^2 = \Lambda_{1u}^2 = \Lambda_{1v}^2$ and $m_e = \rho l$ is the mass of one of the rods.

The preceding results can be used to verify the conditions under which the stability criteria derived in Ref. 9 on the basis of a discrete mathematical model are valid for the distributed model. Indeed, comparing inequalities (61) with inequalities (8–5) of Ref. 9, we conclude that we can be sure of the validity of the stability criteria derived using a model whereby uniform elastic rods are simulated by means of discrete masses, provided the discrete masses are situated at the tip of the rods, where the rods are massless but possessing an equivalent flexural stiffness of such magnitude that the natural frequency for the vibration of the discrete model in the x or y directions is equal to Λ_1. It should be pointed out that, in contrast with Ref. 9, the results derived here are more general, as no assumption concerning the motion of the elastic members is made.

Summary and Conclusions

A new method of approach to the stability problem of hybrid systems, namely systems described by both ordinary and partial differential equations such as the ones encountered in the dynamics of satellites containing distributed elastic parts, is presented. The method consists of an extension of the Liapunov direct method by considering for testing purposes a hybrid form, that is a form which is a Liapunov function and functional at the same time. The method works directly with the hybrid set of differential equations and it involves no modal truncation or spatial discretization. As a result, it is more rigorous than the latter two and it produces sharper stability criteria. The generality of the formulation is enhanced by the use of operator notation to represent certain properties of the distributed elastic medium. Thus, the formulation is applicable to a large variety of elastic members. The complete solution of the eigenvalue problem associated with the elastic members is not needed but only the lowest eigenvalue. For systems which do not admit a closed-form solution of the eigenvalue problem, the lowest eigenvalue may be obtained by an approximate method or experimentally. The present stability method makes use of the fact that Rayleigh's quotient represents an upper bound for the lowest eigenvalue. For stability and instability to be ascertained, the proper sign-definiteness of the Liapunov functional must be established at every point of the elastic domain. Some clues as to the choice of the Liapunov functional are provided. The stability requirements derived by the present method can be expected in general to be more stringent than the ones derived by means of modal truncation or spatial discretization. On the other hand, the present method yields stability criteria which can be accepted with complete confidence, whereas some doubt remains as to the effect of truncation or discretization on the results obtained by means of either of these two processes. Moreover, the amount of work involved in the application of the method presented is not larger, and quite often it is smaller, than the work involved in the application of the other two methods.

As an illustration of the method, the case of a rigid satellite containing flexible antennas in a circular orbit is investigated. If the principal axes of the body when in undeformed position are denoted by x,y,z, then the antennas, simulated by two thin rods, are assumed to lie along the z axis and to be symmetrically distributed with respect to the xy plane. The vibration of the rods is described by two orthogonal components in the xz and yz planes, respectively. The equilibrium position can be identified as the configuration corresponding to gravity-gradient stabilization. The results obtained in this paper are compared with the ones of Ref. 9 which investigates the stability of a related discrete system, namely the system in which each of the two antennas is simulated by one discrete mass. They can be used also to check the validity of stability criteria derived by means of modal truncation, although in the latter case the amount of work involved promises to be substantial.

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